TA: LEE, Yat Long Luca
Email: yllee@math.cuhk.edu.hk
Office: Room 711 AB1 (Temporary), Room 505 AB1 (Until further notice)
Office Hour: Send me an email first, then we will arrange a meeting (if you need it).
Remark: Please let me know if there are typos or mistakes.

Q1

A *finite Fourier series* is of the form

$$a_0 + \sum_{n=1}^{N} (a_n \sin(nx) + b_n \cos(nx))$$

A *trigonometric polynomial* is of the form

$$p(\cos x, \sin x)$$

where $p(x, y)$ is a polynomial with two variables x, y.

Show that a function is a trigonometric polynomial if and only if it is a finite Fourier series.

Solution:
$(\implies)$
This direction follows from mathematical induction. The idea is to show that $\cos^k x \sin^m x$ can be written as a linear combination of $\cos nx$ and $\sin nx$ for all $1 \leq n \leq N$, then we can write the trigonometric polynomial as a finite Fourier series.

$(\impliedby)$
This direction is done similarly as above. We will show by mathematical induction that $\sin nx$ and $\cos nx$ can be written as some sum of $\cos^m x \sin^k x$. Then we will obtain a trigonometric polynomial.

(Details are omitted in this solution)[1]

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[1] When I grade your homework, as long as you can show the result with details, you will have points.

Q2

Let f be a C^∞ 2π-periodic function on $[-\pi, \pi]$. Show that the Fourier coefficients

$$|a_n| = o\left(\frac{1}{n^k}\right) \qquad \text{and} \qquad |b_n| = o\left(\frac{1}{n^k}\right)$$

as $n \to \pm\infty$ for every k.

Solution:
First of all it is smooth, then it is k-differentiable and $f, ..., f^{(k)}$ are all integrable for any $k \in \mathbb{N}$.

Note that for any $k \in \mathbb{N}$, we have[2] $a_n(f^{(k)}) = nb_n(f^{(k-1)})$ and $b_n(f^{(k)}) = -na_n(f^{(k-1)})$, then in general, we have[3]

$$a_n(f^{(k)}) = \begin{cases} (-1)^p n^{2p} a_n(f), & k = 2p \\ (-1)^p n^{2p+1} b_n(f), & k = 2p+1 \end{cases}$$

and

$$b_n(f^{(k)}) = \begin{cases} (-1)^p n^{2p} b_n(f), & k = 2p \\ (-1)^{p+1} n^{2p+1} a_n(f), & k = 2p+1 \end{cases}$$

by Riemann-Lebesgue Lemma that as $n \to \infty$, we have $a_n(f^(k)), b_n(f^(k)) \to 0$ for all $k \in \mathbb{N}$. Then for any $k \in \mathbb{N}$, we have $n^k |a_n f| = |a_n f^{(k)}| \to 0$ as $n \to \infty$. Thus

$$|a_n| = o\left(\frac{1}{n^k}\right)$$

similarly,

$$|b_n| = o\left(\frac{1}{n^k}\right)$$

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[2] $a_n(g)$, $n_n(g)$ denote the Fourier coefficient of any function g.
[3] show it by induction.

Q3

Let f, g and $h \in R[a,b]$. Show that

$$\|f-g\|_2 \leq \|f-h\|_2 + \|h-g\|_2$$

when does the equality sign holds?

Solution:
This is a direct application of the triangle inequality. So, it suffices to show that the triangle inequality

$$\|p+q\|_2 \leq \|p\|_2 + \|q\|_2 \tag{0.1}$$

holds for any $p, q \in R[a,b]$.

Consider $\|p+q\|_2^2 = \int (p+q)^2 = \int p^2 + \int 2pq + \int q^2$. By Cauchy-Schwarz's inequality[4], the middle term can be written as

$$\int pq = \langle p, q\rangle_2 \leq \|p\|_2 \|q\|_2$$

thus

$$\|p+q\|_2^2 = \int p^2 + \int q^2 + 2\int pq \leq \int p^2 + \int q^2 + 2\|p\|_2\|q\|_2 = (\|p\|_2 + \|q\|_2)^2$$

thus, we completed (0.1). Then our desired result follows from

$$\|f-g\|_2 = \|f-h+h-g\|_2 \leq \|f-h\|_2 + \|h-g\|_2$$

Equality holds when $h = \lambda f + (1-\lambda)g$ a.e. for $\lambda \in [0,1]$ as we can check

$$\begin{aligned}\|f-g\|_2 &\leq \|f-\lambda f + (1-\lambda g)\|_2 + \|\lambda f + (1-\lambda)g - g\|_2 \\ &= (1-\lambda)\|f-g\|_2 + \lambda\|f-g\|_2 \\ &= \|f-g\|_2\end{aligned}$$

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[4]exercise

Q4

Let f, g be 2π-periodic functions integrable on $[-\pi, \pi]$. Show that

$$\int_{-\pi}^{\pi} fg = 2\pi a_0(f)a_0(g) + \pi \sum_{n=1}^{\infty} [a_n(f)a_n(g) + b_n(f)b_n(g)]$$

where a_0, a_n, b_n are corresponding Fourier coefficients.

Solution:
We apply the Parseval's identity. Consider

$$\begin{aligned}\|f+g\|_2^2 &= 2\pi a_0(f+g)^2 + \pi \sum_{n=1}^{\infty} \left(a_n(f+g)^2 + b_n(f+g)^2\right) \\ &= 2\pi \left(a_0(f) + a_0(g)\right)^2 + \pi \sum_{n=1}^{\infty} \left(\left(a_n(f) + a_n(g)\right)^2 + \left(b_n(f) + b_n(g)\right)^2\right)\end{aligned}$$

and similarly,

$$\begin{aligned}\|f-g\|_2^2 &= 2\pi a_0(f0g)^2 + \pi \sum_{n=1}^{\infty} \left(a_n(f-g)^2 + b_n(f-g)^2\right) \\ &= 2\pi \left(a_0(f) - a_0(g)\right)^2 + \pi \sum_{n=1}^{\infty} \left(\left(a_n(f) - a_n(g)\right)^2 + \left(b_n(f) - b_n(g)\right)^2\right)\end{aligned}$$

then using the identity $4\int_{-\pi}^{\pi} fg = \|f+g\|_2^2 - \|f-g\|_2^2$ and $4ab = (a+b)^2 - (a-b)^2$, we have

$$\|f+g\|_2^2 - \|f-g\|_2^2 = 8\pi a_0(f)a_0(g) + \sum_{n=1}^{\infty} [4a_n(f)a_n(g) + 4b_n(f) + b_n(g)]$$

thus

$$\int_{-\pi}^{\pi} fg = 2\pi a_0(f)a_0(g) + \pi \sum_{n=1}^{\infty} [a_n(f)a_n(g) + b_n(f) + b_n(g)]$$

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Q5

Using Parseval's Identity for $f(x) = x$ on $[-\pi, \pi]$ to show the famous equality

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Solution:
Consider $f(x) = x$ on $[-\pi, \pi]$, its Fourier series, as computed in lecture 3, is given by

$$f(x) \sim 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx)$$

note, $b_n = \frac{2(-1)^{n+1}}{n}$, then the Parseval's identity tells us

$$\|f\|_2^2 = 2\pi a_0^2 + \pi \sum_{n=1}^{\infty}(a_0^2 + b_0^2) = \sum_{n=1}^{\infty} \frac{4\pi}{n^2}$$

Then we compute

$$\|f\|_2^2 = \int_{-\pi}^{\pi} x^2 \, dx = \frac{2\pi^3}{3}$$

thus

$$\sum_{n=1}^{\infty} \frac{4\pi}{n^2} = \frac{2\pi^3}{3} \implies \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

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